Eur. J. Mech. B/Fluids

A NEW FRAMEWORK FOR STUDYING THE STABILITY OF GENUS-1 AND GENUS-2 KP PATTERNS

Thomas J. Bridges[1]

[1]: *Dept Mathematics and Statistics, Univ. of Surrey, Guildford, Surrey GU2 5XH, England, e-mail:* t.bridges@mcs.surrey.ac.uk

(Received 2 October 1998, revised and accepted 21 December 1998)

Abstract – The Kadomtsev-Petviashvili equation – or KP equation – is a model equation for waves that are weakly two-dimensional in a horizontal plane, and models water waves in shallow water with weak three-dimensionality. It has a vast array of interesting genus$-k$ pattern solutions which can be obtained explicitly in terms of Riemann theta functions. However the linear or nonlinear stability of these patterns has not been studied. In this paper, we present a new formulation of the KP model as a Hamiltonian system on a multi-symplectic structure. While it is well-known that the KP model is Hamiltonian – as an evolution equation in time – multi-symplecticity assigns a distinct symplectic operator for each spatial direction as well, and is independent of the integrability of the equation. The multi-symplectic framework is then used to formulate the linear stability problem for genus-1 and genus-2 patterns of the KP equation; generalizations to genus$-k$ with $k > 2$ are also discussed.

1. Introduction

The Kadomtsev-Petviashvili (KP) equation was one of the first model equations proposed for three-dimensional ocean patterns (two horizontal space dimensions; one vertical direction) (Kadomtsev-Petviashvili,1970). It was first proposed to study the transverse instability of the KdV solitary wave. It was later found to be completely integrable and to have a large class of periodic and quasi-periodic patterns (cf. Dubrovin, 1981 and references therein). In particular there is a large class of genus$-k$ solutions for $k = 1, 2, \ldots$ which can be expressed explicitly in terms of Riemann theta functions. A hypothesis in the derivation of the KP equation is that the waves are primarily uni-directional in a horizontal plane with weak two dimensionality. However comparison of the solutions of the KP with experiments has shown that these patterns are accurate representations of the multi-periodic and quasi-periodic patterns that appear on the ocean surface in shallow water, even at large amplitude and with significant two-dimensionality (cf. Segur & Finkel, 1985, Hammack et al., 1995, Dubrovin et al., 1997 and references therein).

An open question about these genus$-k$ patterns of the KP model is whether they are stable or not as solutions of the model equation. The only results as far as we are aware are for genus-1 patterns of the integrable KP model: in §8.3.1 of Infeld & Rowlands, 1990, a stability analysis of genus-1 patterns is given using an averaged Lagrangian and the Whitham modulation equations. In §3 a different formulation of the stability problem for genus-1 patterns is given that extends to patterns of higher genus. In §4 we present a formulation of the stability problem for genus-2 patterns and the sketch how the theory extends to higher genus patterns.

In this paper we will concentrate on formulating the stability problem. The calculations necessary to evaluate the instability criteria proposed here may be formidable, even when the basic state is known explicitly in terms of theta functions, and this will be considered elsewhere.

The basis for the analysis is a new multi-symplectic formulation of the KP equation and its generalizations. Multi-symplecticity is a framework which assigns a distinct symplectic operator to each space direction and time, and is a natural Hamiltonian structure for pattern formation in conservative systems: symplectic pattern formation (cf. Bridges, 1997a,1997b,1998). Multi-symplecticity is not to be confused with bi-Hamiltonianism or multi-Hamiltonianism which associate more than one symplectic structure in the time direction only, and are associated with integrable systems. A multi-symplectic structure is distinct from integrability; for example, the complete equations governing the water-wave problem have a multi-symplectic structure (cf. Bridges, 1996) and it is unlikely that the water-wave problem is completely integrable except in special cases.

2. Multi-symplectic structure of the KP equation

In this section we show that the KP equation and the generalized KP model can be given a new characterization as a Hamiltonian system on a multi-symplectic structure.

The starting point is the generalized KP equation

$$(2u_t + \partial_x f(u) + u_{xxx})_x + \sigma u_{yy} = 0 \tag{2.1}$$

where σ is a non-zero parameter, $f(u)$ is some smooth function and the 2 multiplying u_t is added for notational convenience (it can be eliminated by scaling t). The KP model relevant for gravity water waves is recovered by taking $\sigma = 3$ and $f(u) = 3u^2$ (cf. Segur & Finkel, 1985: equation (1.8)). The case where $f(u) = u^m$ with m an integer or rational number has been studied extensively by Wang et al., 1994 and de Bouard & Saut, 1997.

Introduce new variables

$$Z = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{R}^4 \times \mathbb{R}^4 . \tag{2.2}$$

These coordinates will be used to reformulate the generalized KP equation as a system of first-order PDEs, and are defined as follows. Define q_1, q_2, q_3 and p_3 by

$$u \stackrel{\text{def}}{=} p_3 = \frac{\partial}{\partial x} q_2 = \frac{\partial^2}{\partial x^2} q_1 \quad \text{and} \quad q_3 = \frac{\partial}{\partial x} p_3 . \tag{2.3}$$

Define p_1, p_2 and p_4 by

$$\begin{aligned} p_1 &= -\frac{\partial p_3}{\partial t} - \frac{\partial p_2}{\partial x} - \sigma \frac{\partial p_4}{\partial y} \\ p_2 &= f(p_3) + \frac{\partial q_2}{\partial t} + \frac{\partial q_3}{\partial x} - \frac{\partial q_4}{\partial y} \\ p_4 &= \frac{\partial q_2}{\partial y} + \frac{1}{\sigma}\frac{\partial q_4}{\partial x} . \end{aligned} \tag{2.4}$$

Then the KP equation (2.1) is equivalent to

$$\frac{\partial p_4}{\partial x} - \frac{\partial p_3}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p_1}{\partial x} = 0 . \tag{2.5}$$

This can be verified by substituting (2.2)-(2.4) into the two equations in (2.5).

Combining these equations leads to a system of eight first-order partial differential equations which can be written in the form

$$\mathbf{M} Z_t + \mathbf{K} Z_x + \mathbf{L} Z_y = \nabla S(Z) , \quad Z \in \mathbb{R}^8 , \tag{2.6}$$

where

$$\mathbf{M}=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},\quad \mathbf{K}=\begin{bmatrix} 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{bmatrix},$$

$$\mathbf{L}=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-\sigma\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&\sigma&0&0&0&0&0&0 \end{bmatrix},$$

and

$$S(Z)=p_1q_2+p_2p_3+\tfrac{1}{2}\sigma p_4^2-\tfrac{1}{2}q_3^2-F(p_3)\,,$$

where

$$F(u)=\int_0^u f(s)\,ds\,.$$

This system is multi-symplectic in the following sense. Each of three skew-symmetric matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ can be identified with closed two forms. Let

$$\omega^{(1)}=\mathbf{d}p_3\wedge\mathbf{d}q_2\,,\quad \omega^{(2)}=\sum_{j=1}^{4}\mathbf{d}p_j\wedge\mathbf{d}q_j\quad\text{and}\quad \omega^{(3)}=\mathbf{d}q_4\wedge\mathbf{d}p_3+\sigma\mathbf{d}p_4\wedge\mathbf{d}q_2\,.$$

Then the three matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ are defined by

$$\omega^{(1)}(U,V)=\langle\mathbf{M}U,V\rangle\,,\quad \omega^{(2)}(U,V)=\langle\mathbf{K}U,V\rangle\quad\text{and}\quad \omega^{(3)}(U,V)=\langle\mathbf{L}U,V\rangle\,,$$

where U,V are any vectors in $\mathbb{R}^8$ and $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product on $\mathbb{R}^8$.

Each of the two forms $\omega^{(j)}$ $j=1,2,3$ is closed (since they are constant) and therefore they are pre-symplectic forms on $\mathbb{R}^8$, and on subspaces where they are non-degenerate, they are symplectic forms. In other words:

$$(\mathbb{R}^2,\omega^{(1)})\,,\quad (\mathbb{R}^8,\omega^{(2)})\quad\text{and}\quad (\mathbb{R}^4,\omega^{(3)})$$

are three distinct symplectic manifolds. Moreover, each is associated with a different direction: $\omega^{(1)}$ is associated with time; $\omega^{(2)}$ is associated with the x-direction and $\omega^{(3)}$ is associated with the y-direction. Because of the ordering of the coordinates, the two form $\omega^{(2)}$ is in standard form for a symplectic form on $\mathbb{R}^8$.

A significant aspect of the formulation (2.6) is that the phase space is *finite*-dimensional, whereas a classical Hamiltonian formulation in the time direction only requires specification of a function space and involves an *infinite*-dimensional phase space. It is the abstract form of the equations (2.6) that is the basis for the present analysis. In other words the KP equation and the generalized KP model are completely characterized by the function $S(Z)$, and the three skew-symmetric operators $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ – all defined on a finite-dimensional space.

Note that multi-symplecticity is a different concept from bi-Hamiltonianism or multi-Hamiltonianism. Bi-Hamiltonian structures associate two distinct symplectic structures with the time direction only and are a precursor to integrability. On the other hand, multi-symplecticity associates a distinct symplectic structure with each independent space and time direction and is independent of integrability of the PDE. Multi-symplecticity is a natural dynamical systems framework for pattern formation in conservative systems (cf. Bridges, 1996,1997a,1997b,1998).

3. Genus-1 patterns

A genus-1 pattern of the KP equation is a solution of (2.6) of the form

$$Z(x,y,t) = \widehat{Z}(\theta) \quad \text{with} \quad \theta = \kappa x + \ell y + \omega t + \theta^0 . \tag{3.1}$$

Substitution of this form into (2.6) results in

$$[\omega \mathbf{M} + \kappa \mathbf{K} + \ell \mathbf{L}]\widehat{Z}_\theta = \nabla S(\widehat{Z}) , \tag{3.2}$$

or

$$\mathbf{J}\widehat{Z}_\theta = \nabla S(\widehat{Z}) , \quad \text{with} \quad \mathbf{J} = \omega \mathbf{M} + \kappa \mathbf{K} + \ell \mathbf{L} . \tag{3.3}$$

When $\kappa \neq 0$ the skew-symmetric matrix $\mathbf{J}$ is non-degenerate; hence the system (3.3) can be viewed as a classical Hamiltonian system on $\mathbb{R}^8$, and periodic travelling waves correspond to periodic orbits of this Hamiltonian system.

Another view of solutions of (3.2)-(3.3) which we will find useful is as critical points of a constrained variational principle. Since $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{L}$ are skew-symmetric, the operators $\mathbf{M}\partial_\theta$, $\mathbf{K}\partial_\theta$ and $\mathbf{L}\partial_\theta$ are symmetric operators on a space of periodic functions and therefore each can be characterized as the gradient of a functional. Let

$$\mathcal{A}(Z) = \tfrac{1}{2} \oint \langle \mathbf{M}Z_\theta , Z \rangle \, d\theta , \quad \mathcal{B}(Z) = \tfrac{1}{2} \oint \langle \mathbf{K}Z_\theta , Z \rangle \, d\theta , \quad \mathcal{C}(Z) = \tfrac{1}{2} \oint \langle \mathbf{L}Z_\theta , Z \rangle \, d\theta , \tag{3.4a}$$

where $\oint d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta$; equivalently,

$$\mathcal{A}(Z) = \tfrac{1}{2} \oint \omega^{(1)}(Z_\theta , Z) \, d\theta , \; \mathcal{B}(Z) = \tfrac{1}{2} \oint \omega^{(2)}(Z_\theta , Z) \, d\theta , \; \mathcal{C}(Z) = \tfrac{1}{2} \oint \omega^{(3)}(Z_\theta , Z) \, d\theta . \tag{3.4b}$$

Define the set

$$\mathcal{I}(I) = \{ \; Z \in C^1(S^1, \mathbb{R}^8) \; : \; \mathcal{A}(Z) = I_1 \, , \; \mathcal{B}(Z) = I_2 \text{ and } \mathcal{C}(Z) = I_3 \, , \quad I \in \mathbb{R}^3 \; \} , \tag{3.5}$$

where $C^1(S^1, \mathbb{R}^8)$ is the space of all continuously differentiable $2\pi-$periodic eight-component functions. Then genus-1 patterns can be characterized as critical points of the functional $\mathcal{S} = \oint S \, d\theta$ restricted to the set $\mathcal{I}(I)$, with ω, κ and ℓ as Lagrange multipliers. The Lagrange functional is

$$\mathcal{F}(Z, \omega, \kappa, \ell) = \mathcal{S}(Z) - \omega \mathcal{A}(Z) - \kappa \mathcal{B}(Z) - \ell \mathcal{C}(Z) . \tag{3.6}$$

With the values of the constraint sets considered specified, the critical point, $\widehat{Z}(\theta; I_1, I_2, I_3)$, is a function of θ as well as the parameters I_1, I_2, I_3.

Classical Lagrange multiplier theory then leads to the identities

$$\omega = \frac{\partial \mathcal{S}}{\partial I_1} , \quad \kappa = \frac{\partial \mathcal{S}}{\partial I_2} , \quad \ell = \frac{\partial \mathcal{S}}{\partial I_3} . \tag{3.7}$$

The constrained variational principle is said to be non-degenerate precisely when

$$\det \begin{bmatrix} \frac{\partial\omega}{\partial I_1} & \frac{\partial\omega}{\partial I_2} & \frac{\partial\omega}{\partial I_3} \\ \frac{\partial\kappa}{\partial I_1} & \frac{\partial\kappa}{\partial I_2} & \frac{\partial\kappa}{\partial I_3} \\ \frac{\partial\ell}{\partial I_1} & \frac{\partial\ell}{\partial I_2} & \frac{\partial\ell}{\partial I_3} \end{bmatrix} \neq 0\,. \tag{3.8}$$

Using the identities (3.7) the non-degeneracy condition has the equivalent form

$$\det[\mathrm{Hess}_I(\mathcal{S})] = \det \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ \mathcal{S}_{21} & \mathcal{S}_{22} & \mathcal{S}_{23} \\ \mathcal{S}_{31} & \mathcal{S}_{32} & \mathcal{S}_{33} \end{bmatrix} \neq 0\,, \quad \text{where} \quad \mathcal{S}_{ij} = \frac{\partial^2 \mathcal{S}}{\partial I_i \partial I_j}\,. \tag{3.9}$$

This system is now in standard form to apply the instability theory for periodic travelling waves in (Bridges, 1997a, §4). Theorem 4.1 and Corollary 4.2 give conditions on the entries of the matrix in (3.9) for instability. This theory applies whether the original equation is integrable or not. However, when the system is integrable, for example when $f(u)$ is a quadratic polynomial, explicit solutions for these waves can be written down and therefore the entries of the matrices (3.8) and (3.9) can be explicitly constructed.

One issue that has not been considered here, which may effect the stability result, is the role of meanflow. It is well-known that mean flow effects in oceanography can stabilize an otherwise unstable wavetrain; an example of this is the stabilization of the Benjamin-Feir instability as the wave travels into shallow water (see Whitham, 1974, §16.9 for a discussion of the effect of meanflow on stability of the Stokes wave).

The KP model for weakly 3D gravity waves in shallow water (i.e. the KP model with $\sigma = 3$ and $f(u) = 3u^2$) has a restricted form of meanflow. In the multi-symplectic framework, meanflow can be characterized as flow along a group orbit. In other words, to include meanflow effects, it is important to identify the symmetries and conservation laws of the KP model and include them in the construction and linear stability analysis of genus−1 KP patterns. Examples of how meanflow can be incorporated into a multi-symplectic framework – and affect stability – are in (Bridges, 1995) and (Bridges, 1996).

4. Genus-2 patterns

A genus−2 pattern of the KP equation is a solution of (2.6) of the form

$$Z(x,y,t) = \widehat{Z}(\theta_1,\theta_2) \quad \text{with} \quad \theta_j = \kappa_j x + \ell_j y + \omega_j t + \theta_j^0\,, \quad j = 1,2\,. \tag{4.1}$$

Substitution of this form into (2.6) results in

$$\mathbf{J}_1 \frac{\partial \widehat{Z}}{\partial \theta_1} + \mathbf{J}_2 \frac{\partial \widehat{Z}}{\partial \theta_2} = \nabla S(\widehat{Z})\,, \quad \text{where} \quad \mathbf{J}_j = \omega_j \mathbf{M} + \kappa_j \mathbf{K} + \ell_j \mathbf{L}\,. \tag{4.2}$$

Although the skew-symmetric matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ are non-degenerate when κ_1 and κ_2 are non-zero, the system (4.2) is no longer a classical Hamiltonian system. However it retains the multi-symplectic structure and the variational principle for genus−1 patterns carries over in a straightforward way.

For $j = 1, 2$, let

$$\mathcal{A}_j(Z) = \tfrac{1}{2} \oint \omega^{(1)}(Z_{\theta_j}, Z)\, d\theta\,,\ \mathcal{B}_j(Z) = \tfrac{1}{2} \oint \omega^{(2)}(Z_{\theta_j}, Z)\, d\theta\,,\ \mathcal{C}_j(Z) = \tfrac{1}{2} \oint \omega^{(3)}(Z_{\theta_j}, Z)\, d\theta\,, \tag{4.3}$$

where

$$\oint d\theta = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} d\theta_1\, d\theta_2\,.$$

Define the set

$$\mathcal{I}(I) = \{\ Z \in C^1(\mathbb{T}^2, \mathbb{R}^8)\ :\ \mathcal{A}_j(Z) = I_{3j-2}\,,\ \mathcal{B}(Z) = I_{3j-1} \text{ and } \mathcal{C}(Z) = I_{3j}\ \}\,, \tag{4.4}$$

where $C^1(\mathbb{T}^2, \mathbb{R}^8)$ is the set of all continuously differentiable $2\pi \times 2\pi$−periodic eight-component functions and

$$I = I_1, \dots, I_6 \in \mathbb{R}^6\,.$$

Then genus−2 patterns can be characterized as critical points of the functional $\mathcal{S} = \oint S\, d\theta$ restricted to the set $\mathcal{I}(I)$, with ω_j, κ_j and ℓ_j, $j = 1,2$ as Lagrange multipliers. The Lagrange functional is

$$\mathcal{F}(Z, \omega, \kappa, \ell) = \mathcal{S}(Z) - \sum_{j=1}^{2} (\omega_j\, \mathcal{A}_j(Z) + \kappa_j\, \mathcal{B}_j(Z) + \ell_j\, \mathcal{C}_j(Z))\,. \tag{4.5}$$

With the values of the constraint sets considered specified, the critical point, $\widehat{Z}(\theta; I_1, \dots, I_6)$, is a function of θ_1, θ_2 as well as the parameters $I_1, \dots, I_6$.

Classical Lagrange multiplier then leads to the identities

$$\omega_j = \frac{\partial \mathcal{S}}{\partial I_{3j-2}}\,, \quad \kappa_j = \frac{\partial \mathcal{S}}{\partial I_{3j-1}}\,, \quad \ell_j = \frac{\partial \mathcal{S}}{\partial I_{3j}}\,, \quad j = 1, 2\,. \tag{4.6}$$

The constrained variational principle is said to be non-degenerate precisely when

$$\det \begin{bmatrix} \frac{\partial \omega_1}{\partial I_1} & \cdots & \frac{\partial \omega_1}{\partial I_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial \ell_2}{\partial I_1} & \cdots & \frac{\partial \ell_2}{\partial I_6} \end{bmatrix} \neq 0\,. \tag{4.7}$$

Using the identities (4.6) the non-degeneracy condition has the equivalent form

$$\det \begin{bmatrix} \mathcal{S}_{11} & \cdots & \mathcal{S}_{16} \\ \vdots & \ddots & \vdots \\ \mathcal{S}_{61} & \cdots & \mathcal{S}_{66} \end{bmatrix} \neq 0\,, \quad \text{where } \mathcal{S}_{ij} = \frac{\partial^2 \mathcal{S}}{\partial I_i \partial I_j}\,. \tag{4.8}$$

This variational principle holds for any genus−2 solution of a system of the form (2.6), and is therefore independent of integrability of the PDE. According to the variational principle, a genus−2 pattern is specified by the eight parameters: two phases θ_1^o and θ_2^o in (4.1) and values $I_1, \dots, I_6$ of the six constraint sets. Alternatively, the six Lagrange multipliers could be fixed. In general the twelve parameters $I_1, \dots, I_6, \omega_1, \dots, \ell_2$ are arranged in dual pairs:

$$(I_1, \omega_1)\,, \quad (I_2, \omega_2)\,, \quad (I_3, \kappa_1)\,, \quad (I_4, \kappa_2)\,, \quad (I_5, \ell_1)\,, \quad (I_6, \ell_2)\,.$$

In the specification of a genus−2 pattern, one of each pair can be fixed (with the other determined by the solution).

When $f(u)$ is a quadratic function of u, the KP equation is completely integrable and the genus−2 patterns can be expressed explicitly in terms of theta functions (cf. Dubrovin, 1981, Segur and Finkel, 1985). In this case, the genus−2 pattern is specified in terms of a Riemann matrix and the phases and is non-degenerate when a certain 4×4 matrix (denoted by $\mathbf{M}$ in (Segur and Finkel, 1985, equation (4.12))) is non-degenerate (Dubrovin, 1981), (Segur and Finkel, 1985). We conjecture that – in the integrable case – the non-degeneracy condition of Dubrovin and the condition (4.6) are equivalent.

The stability of genus−2 patterns can be formulated as follows. Let

$$Z(x, y, t) = \widehat{Z}(\theta_1, \theta_2; I) + \widehat{V}(\theta_1, \theta_2, t)\,,$$

and substitute into (2.6) and linearize about $\widehat{Z}$:

$$\mathbf{M}\widehat{V}_t + \mathbf{J}_1\widehat{V}_x + \mathbf{J}_2\widehat{V}_y = D^2S(\widehat{Z})\widehat{V}\,. \tag{4.9}$$

With a spectral ansatz: $\widehat{V}(\theta_1,\theta_2,t) = e^{\lambda t}V(\theta_1,\theta_2,t)$, the system (4.9) is reduced to the spectral problem

$$\mathbf{J}_1V_x + \mathbf{J}_2V_y = D^2S(\widehat{Z})V - \lambda\mathbf{M}V\,. \tag{4.10}$$

The analysis of this spectral problem is difficult because the coefficients are doubly periodic (i.e. toral) and the operator may not have closed range (see discussion in Bridges, 1998). Therefore we proceed formally and make the "Floquet ansatz"

$$V(\theta_1,\theta_2,\lambda) = e^{i(\alpha_1\theta_1+\alpha_2\theta_2)}U(\theta_1,\theta_2,\lambda,\alpha_1,\alpha_2)\,.$$

Then the function U satisfies

$$\mathcal{L}U = \lambda\mathbf{M}U + i\alpha_1\mathbf{J}_1U + i\alpha_2\mathbf{J}_2U\,, \tag{4.11}$$

with

$$\mathcal{L} = D^2S(\widehat{Z}) - \mathbf{J}_1\frac{\partial}{\partial\theta_1} - \mathbf{J}_2\frac{\partial}{\partial\theta_2}\,. \tag{4.12}$$

The system (4.11) is the basis for the stability theory. Formally, if there exists a $2\pi\times2\pi$−periodic function U satisfying (4.11) for some $(\alpha_1,\alpha_2)\in\mathbb{R}^2$ with $\lambda\in\mathbb{C}$ and $\mathrm{Re}(\lambda)>0$, we say that the genus−2 patterns are linearly unstable. An instability result for $|\alpha_1|^2+|\alpha_2|^2 << 1$ can be obtained by noting that $\partial_{\theta_1}\widehat{Z}$ and $\partial_{\theta_2}\widehat{Z}$ are in the kernel of $\mathcal{L}$. If we assume that the kernel of $\mathcal{L}$ is not larger (i.e. we neglect KP meanflow effects and assume that the kernel is not degenerate) then the right-hand side of (4.11) can be projected onto the kernel of $\mathcal{L}$ leading to a dispersion relation for $(\lambda,\alpha_1,\alpha_2)$ when these parameters are sufficiently small. An analysis of this type for multi-phase patterns of arbitrary systems with a multi-symplectic structure is given in (Bridges, 1998). The full details of the application of this theory to the KP model will be given elsewhere.

5. Concluding Remarks

In this paper, aspects of the formulation of the stability problem for genus−1 and genus−2 patterns of the KP equation and the generalized KP model were presented. The starting point for the analysis was a new formulation of the KP model as a Hamiltonian system on a multi-symplectic structure. An advantage of the multi-symplectic formulation is that abstract results can be applied to deduce sufficient conditions for instability, and new instability criteria can be deduced based on the structural properties of the equations alone.

While there some results in the literature on the stability of genus−1 patterns, even for the full water-wave problem, and results on the stability of solitary waves (cf. Wang et al., 1994 and de Bouard and Saut, 1997), the stability of genus−2 patterns of the KP equation is an open problem. The theory of §4 coupled with an explicit form for the genus−2 pattern provides a relatively straightforward framework for checking instability for $|\alpha_1|$ and $|\alpha_2|$ small.

The analysis of §4 formally extends with minor modification to genus−k patterns for k any (finite) natural number. In this case there are k phases of the form (4.1) and the left-hand side of (4.2) has k terms. The variational principle with Lagrange functional (4.5) also extends in a straightforward way. There are k phases and $3k$ constraint sets for a total of $4k$ parameters specifying the genus−k pattern. The stability framework also extends, but in general one can expect small divisors. The main problem with small divisors is that the operator $\mathcal{L}$ in §4 (and its generalization for $k>2$) does not have close range and so the Fredholm alternative does not rigorously apply. On the other hand, when the system is completely integrable, we conjecture that the small divisor issue can be eliminated and then the analysis of the stability equation (4.11) (and its generalization to genus−k patterns) is exact and rigorous.

References

[1] DE BOUARD, A., SAUT, J. C., 1997 *Symmetries and decay of the generalized Kadomtsev-Petviashvili solitary waves*, SIAM J. Math. Anal. **28**, 1064-1085.

[2] BRIDGES, T. J., 1995 *Multi-symplectic structures, Boussinesq equations and periodic travelling waves*, Proceedings of the IUTAM Symposium on Nonlinear Waves in Fluids, (EDS. A. MIELKE & K. KIRCHGÄSSNER), World Scientific: Singapore.

[3] BRIDGES, T. J., 1996 *Periodic patterns, linear instability, symplectic structure and mean-flow dynamics for three-dimensional surface waves*, Phil. Trans. Roy. Soc. Lond. **A354**, 533-74.

[4] BRIDGES, T. J., 1997a *Multi-symplectic structures and wave propagation*, Math. Proc. Camb. Phil. Soc. **121**, 147-190.

[5] BRIDGES, T. J., 1997b *A geometric formulation of the conservation of wave action and its implications for signature and the classification of instabilities*, Proc. Roy. Soc. Lond. **A453**, 1365-1395.

[6] BRIDGES, T. J., 1998 *Toral-equivariant partial differential equations and quasiperiodic patterns*, Nonlinearity **11**, 467-500.

[7] DUBROVIN, B. A., 1981 *Theta functions and nonlinear equations*, Russ. Math. Surveys **36**, 11-92.

[8] DUBROVIN, B. A., FLICKINGER, R., SEGUR, H., 1997 *Three-phase solutions of the Kadomtsev-Petviashvili equation*, Stud. Appl. Math. **99**, 137-203.

[9] HAMMACK, J., MCCALLISTER, D., SCHEFFNER, N., SEGUR, H., 1995 *Two-dimensional periodic waves in shallow water. Part 2. Asymmetric waves*, J. Fluid Mech. **285**, 95-122.

[10] KADOMTSEV, B. B., PETVIASHVILI, V. I., 1970 *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. **15**, 539-541.

[11] INFELD, I., ROWLANDS, G., 1990 *Nonlinear Waves, Solitons and Chaos*, Cambridge University Press.

[12] SEGUR, H., FINKEL, A., 1985 *An analytical model of periodic waves in shallow water*, Stud. Appl. Math. **73**, 183-220.

[13] WHITHAM, G. B., 1974 *Linear and Nonlinear Waves*, Wiley-Interscience Publishers: New York.

[14] WANG, X. P., ABLOWITZ, M. J., SEGUR, H., 1994 *Wave collapse and instability of solitary waves of a generalized Kadomtsev-Petviashvili equation*, PhysicaD **78**, 241-265.